JOURNAL OF APPROXIMATION THEORY **14**, 30–37 (1975)

Weak Chebyshev Sets and Splines

M. W. BARTELT

Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York 12181

Communicated by G. G. Lorentz

1. INTRODUCTION

Let M denote an n dimensional subspace of $C[a, b]$, the space of continuous real valued functions on the compact interval $[a, b]$. Then M is said to be a Haar (weak Chebyshev) set if each function ($\neq 0$) in M has no more than $n - 1$ zeroes (sign changes) on $[a, b]$.

This note has two related goals. The first is to clarify the relationship between Haar sets and weak Chebyshev sets. It is shown in Theorem 1 that any weak Chebyshev set M containing the constants is a Haar set if and only if each nonzero function in M has finitely many zeroes. In [1] a study was made of weak Chebyshev sets. Splines are an example of a weak Chebyshev set [1, 2]. The second goal is to obtain in Theorem 3, a partial converse to this result and in Theorem 4, a generalization of this result. The weak Chebyshev character of splines gives rise to alternation properties of the best approximate [1, 3, 4].

2. WEAK CHEBYSHEV AND HAAR SETS

We first give some sufficient conditions under which a weak Chebyshev set is in fact a Haar set.

As usual for $f \in C[a, b]$ we say that x is a simple zero if $f(x) = 0$ and f changes sign at x and x is a $(+)$ double zero if $f(x) = 0$ and $f \geq 0$ near x. A $(-)$ double zero is similarly defined.

LEMMA 1. *Let M be an n dimensional subspace of $C[a, b]$ with $1 \in M$. If there exists an $m \in M$ with precisely N zeroes, counting double zeroes twice, then there exists an $m \in M$ with at least N sign changes.*

Proof. Let $m \in M$ have N zeroes, $x_1, x_2, \ldots, x_N$. If m changes sign at each x_i we are done. If not, then m has some double zeroes. Assume without loss of generality that f has at least as many $(+)$ double zeroes as $(-)$ double zeroes. If x is a (+) double zero, then for some $\epsilon > 0$, $f - \epsilon$ has two zeroes near x. Also if x is a simple zero of f, then there is an $\epsilon > 0$ such that $f - \epsilon$ has a zero arbitrarily close to x. Now let:

$x_1, x_2, \dots, x_j$ be the simple zeroes of f,

$x_{j+1}, x_{j+2}, \dots, x_{j+k}$ the (+) double zeroes of f, and

$x_{j+k+1}, x_{j+k+2}, \dots, x_N$ the (−) double zeroes of f.

Then there is an $\epsilon > 0$ such that $f - \epsilon$ has a zero at points $x_1', x_2', \dots, x_j'$, near the simple zeroes of f, and two zeroes near the (+) double zeroes of f. Thus $f - \epsilon$ has $j + 2k \geq N$ zeroes.

THEOREM 1. *Let M be an n dimensional weak Chebyshev subspace of $C[a, b]$ with $1 \in M$. If each $m \in M$ ($\neq 0$) has finitely many zeroes, then M is a Haar set on $[a, b]$.*

Proof. Follows immediately from Lemma 1.

LEMMA 2. *Let M be an n-dimensional subspace of $C[a, b]$ with $1 \in M$. Assume that there exists an integer N such that each $m \in M$ has fewer then N sign changes. Suppose that there exists a nonzero $m \in M$ with infinitely many zeroes, then there exists an $m \in M$ with $m \equiv 0$ on a subinterval of $[a, b]$.*

Proof. There exists a sequence $\{x_n\}$, $n = 1, 2, \dots$, such that $m(x_n) = 0$, $n = 1, 2, \dots$, and $\{x_n\}$ converges to x_0 in $[a, b]$. Assume without loss of generality that $\{x_n\} \uparrow x_0$. Now by hypothesis m cannot change sign at each successive x_n so eventually m must satisfy $m \geq 0$, or $m \leq 0$, between each x_n. But then for some constant c, $m - c$ has more than N sign changes unless $m \equiv 0$ near x_0. Thus $m \equiv 0$ near x_0.

We then immediately obtain a partial classification of weak Chebyshev sets.

THEOREM 2. *Let M be a weak Chebyshev set on $[a, b]$ with $1 \in M$. Then one and only one of the following occurs:*

(i) *M is a Haar set.*

(ii) *there exists a nonzero $m \in M$ with $m \equiv 0$ on a subinterval of $[a, b]$.*

It should be observed that neither Theorem 1 nor Theorem 2 holds without the assumption $1 \in M$, as can be easily verified by considering the weak Chebyshev subspace of $C[0, 1]$ given by $M = \langle x, x^2, \dots, x^n \rangle$.

3. Weak Chebyshev Sets and Splines

Polynomial splines have been given [1, 2] as an example of a weak Chebyshev set. To be specific, the cubic splines on $[a, b]$ with knots $a = x_0 < x_1 < \cdots < x_n = b$ are the functions in $C^2[a, b]$ whose restriction to any subinterval $[x_i, x_{i+1}]$, $i = 0, 1,..., n - 1$ is a polynomial of degree at most three.

Obviously not all weak Chebyshev sets are obtained in this manner by "piecing" together polynomials. However, any weak Chebyshev set, under appropriate hypotheses, is obtained by "piecing" together Haar sets. To show this we first give two lemmas.

Lemma 3. *Let M be a finite dimensional subspace of $C[a, b]$. Suppose that there exists a sequence $\{f_n\}$, $n = 1, 2,...$ in M with $f_n \equiv 0$ on $[a_n, b]$, where $a < a_n \leq b$ and $f_n \not\equiv 0$ on $[a, b]$. Then*

$$a' = \inf_n a_n ,$$

satisfies $a' > a$. Furthermore, there exists an $m \in M$ with $m = 0$ on $[a', b]$ and $m \not\equiv 0$ on $[a, b]$.

Proof. Assume without loss of generality that $\{a_n\} \downarrow a'$ and let $g_n = f_n / \| f_n \|$, $n = 1, 2,...,$ where

$$\| f_n \| = \sup\{ | f(x)|: x \in [a, b]\}.$$

Then since the unit ball of M is compact in the Chebyshev norm, we can assume without loss of generality that $\{g_n\}$ converges uniformly to g with $\| g \| = 1$. Since the convergence is uniform on $[a, b]$, clearly $g = 0$ on $[a', b]$ and $a' > a$.

Lemma 4. *Let M be a finite dimensional subspace of $C[a, b]$. Suppose that there exists a sequence $\{f_n\}$, $n = 1, 2,...$ in M such that*

(i) *$f_n = 0$ on $[b_n, b]$ where $a < b_n \leq b$ and*

(ii) *$f_n \not\equiv 0$ on any interval $[b_n', b]$ with $b_n' < b_n$.*

Then

$$b' = \sup_n b_n ,$$

satisfies $b' < b$. Furthermore there exists an $m \in M$ with $m = 0$ on $[b', b]$ and $m \not\equiv 0$ on any subinterval of $[a, b]$ properly containing $[b', b]$.

Proof. Let N be the dimension of M. Assume that such sequences $\{f_n\}$ and $\{b_n\}$ exist for $n = 1, 2,..., N+1$ with $b_1 < b_2 < \cdots < b_{N+1}$. Then it follows that $\{f_n\}$, $n = 1, 2,..., N+1$ are linearly independent and this contradiction establishes the result. Indeed, assume that

$$\sum_{i=1}^{N+1} a_i f_i(x) = 0, \qquad \forall\, x \in [a, b].$$

Then there exists an $x_{N+1} \in [b_N, b_{N+1}]$ such that

(i) $f_{N+1}(x_{N+1}) \neq 0$,

(ii) $f_j(x_{N+1}) = 0, j = 1, 2,..., N$.

Thus $a_{N+1} = 0$. Similarly $a_N = a_{N-1} = \cdots = a_1 = 0$. Since there can be only finitely many different b_n's, the existence of the required $m \in M$ follows.

Of course this lemma holds on the other end of the interval i.e., where $f_n \equiv 0$ on $[a, a_n]$, etc.

As usual for $[c, d] \subset [a, b]$, let $M/[c, d]$ denote the restriction of the functions in M to the subset $[c, d]$.

THEOREM 3. *Let M be a finite dimensional weak Chebyshev set on $[a, b]$ with $1 \in M$. Assume also that there exists a $\delta > 0$ such that if $m \in M$ and $m \equiv 0$ on $[c, d] \subset [a, b]$, then $d - c \geqslant \delta$.*

Then there exist knots $a = x_0 < x_1 < \cdots < x_s = b$ such that $M/[x_i, x_{i+1}]$ is a Haar set, $i = 0, 1,..., s-1$.

Proof. Assume that we have chosen as a knot the point

$$y_1 = a + k\delta \qquad \text{with} \quad a + (k+1)\,\delta \leqslant b.$$

We show how a finite sequence of knots $y_2, y_3,..., y_N$ may be found such that M restricted to the intervals $[y_j, y_{j+1}]$ for $j = 1, 2,..., N-1$ and to $[y_N, y_1 + \delta]$ is a Haar set.

Let

$$M(y_1) = \{m \in M: m/[y_1, y_1+\delta] \not\equiv 0,\ m \equiv 0 \text{ on some subinterval in } [y_1, y_1+\delta]\}.$$

If $M(y_1)$ is empty, then M is a Haar set on $[y_1, y_1 + \delta]$ by Theorem 2. For a given subinterval $[c, d]$ in $[y_1, y_1 + \delta]$ on which a given $m \in M(y_1)$ is identically zero, let $[a_m, b_m]$ denote the largest interval in $[a, b]$ containing $[c, d]$ such that $m \equiv 0$ on $[a_m, b_m]$. Then either

$$a_m < y_1 < b_m < y_1 + \delta \qquad \text{or} \qquad y_1 < a_m < y_1 + \delta < b_m.$$

Let

$$B_1 = \inf_m \{b_m : m \in M(y_1),\ a_m < y_1\} \qquad \text{and}$$
$$A_1 = \inf_m \{a_m : m \in M(y_1),\ a_m > y_1\} \qquad \text{and}$$
$$y_2 = \min\{A_1, B_1\}.$$

Then by Lemma 4, $B_1 > y_1$ and by Lemma 3, $A_1 > y_1$. Hence $y_2 > y_1$. Now since no nonzero function in $M/[y_1, y_2]$ can have infinitely many zeroes on $[y_1, y_2]$ by Lemma 2 and the construction of y_2, Theorem 2 implies that $M/[y_1, y_2]$ is a Haar set.

Now for $n = 2, 3, \ldots$, let

$$M(y_n) = \{m \in M : m/[y_n, y_1 + \delta] \neq 0,\ m = 0 \text{ on a subinterval of } [y_n, y_1 + \delta]\}$$

and define A_n and B_n as before. In this manner one obtains $y_1 < y_2 < \cdots < y_1 + \delta$. If infinitely many distinct such y_j's were so obtained, then since $y_j = \min\{A_j, B_j\}$, it would follow that for infinitely many integers j, $y_j = A_j$ or for infinitely many integers j, $y_j = B_j$. Suppose the former case occurred and $y_j = A_j$ for $j \in J$, where J is a set of infinitely many integers. Let $y' = \sup\{y_j : j \in J\}$. Then $y' \leq y_1 + \delta$ and y' is not one of these y_j's. Also there exists for each $j \in J$, a function $m(j)$ identically zero on $[a_{m(j)}, b_{m(j)}]$ such that $a_{m(j)} = y_j$. But with y' playing the role of b in Lemma 4, we see that $y' > \sup\{y_j : j \in J\}$. This contradicts the definition of y' and thus the former case can not occur.

Suppose the latter case occurred and $y_j = B_j$ for $j \in J$, where J is a set of infinitely many integers. Then for each $j \in J$, there is a function $m(j)$, such that $a_{m(j)} < y_1 < b_{m(j)} < y_1 + \delta$ and $B_j = b_{m(j)}$. Let $y' = \sup\{y_j : j \in J\}$. Then y' is not one of the y_j's. Applying Lemma 3 to the intervals $[b_{m(j)}, y']$, we see that $\sup\{b_{m(j)} : j \in J\} < y'$. This contradicts the definition of y' and thus the latter case can not occur.

Thus the process terminates after finitely many steps at $y_N < y_1 + \delta$. Then just as for $M/[y_1, y_2]$, we have that $M/[y_j, y_{j+1}]$, $j = 2, 3, \ldots, N-1$ and $M/[y_N, y_1 + \delta]$ are Haar sets.

Now if $b = a + L\delta$ for some integer L we are done. If not then one eventually obtains a knot,

$$z_1 = a + L\delta \qquad \text{with} \qquad a + (L+1)\delta > b.$$

Let

$$M(z_1) = \{m \in M : m/[z_1, b] \neq 0 \text{ on some subinterval of } [z_1, b]\}.$$

Let $[a_m, b_m]$ be defined as before. Then $a_m < z_1$ since $b - (a + L\delta) < \delta$.

Let B_1 be defined as before and let $z_2 = B_1$. Then $z_2 > z_1$ by Lemma 4. Continuing the process one obtains $z_1 < z_2 < \cdots$, and by Lemma 3 the process terminates at $z_N < b$.

It should be observed that Theorem 3 is not valid without the assumption on the lengths of the intervals where functions in M are identically zero as is shown by taking $M = \langle 1, m\rangle$ in $C[0, 5/12]$ where,

$$m(x) = 1/n, \; 1/(n+1) + 1/3n^2 \leqslant x \leqslant 1/n - 1/3n^2,$$

for $n = 2, 3,\ldots$ and $m(x)$ is linear otherwise.

Theorem 3 also does not hold unless $1 \in M$ as the simple example $M = \langle x\rangle$ on $[0, 1]$ shows.

The following Theorem is a generalization of the result of S. Karlin and W. J. Studden that the polynomial splines are a weak Chebyshev set. It is also a converse to Theorem 3.

THEOREM 4. *Let $a = x_0 < x_1 < \cdots < x_s = b$ be knots on $[a, b]$. For $i = 1, 2,\ldots, s$, let M_i be Haar sets of dimension n_i on the intervals $[x_{i-1}, x_i]$, and assume $1 \in M_i$. Let*

$$M = \{m \in C[a, b]\colon m/[x_{i-1}, x_i] \in M_i, \; i = 1, 2,\ldots, s\}.$$

Then M is a weak Chebyshev set on $[a, b]$ of dimension, $\sum_{i=1}^{m} n_i - (s-1)$.

Proof. First consider the case $s = 2$. Then M consists of those functions $m \in C[a, b]$ such that

$$m/[a, x_1] = p \in M_1 \qquad \text{and} \qquad m/[x_1, b] = q \in M_2.$$

Let $p * q$ denote this function m and letting $m_2 = 1$, let

$$M_1 = \{1, m_2,\ldots, m_{n_1}\}$$

and

$$M_2 = \{1, m_{n_1+1},\ldots, m_{n_1+n_2-1}\}.$$

Then the following is a basis for M;

$$m_i * c_i, \qquad i = 1, 2,\ldots, n_1,$$

$$c_i * m_i, \qquad i = n_1 + 1,\ldots, n_1 + n_2 - 1,$$

where $m_i(x_1) = c_i$, $i = 1, 2,\ldots, n_1 + n_2 - 1$. To verify this observe that if there are constants α_i, $i = 1, 2,\ldots, n_1 + n_2 - 1$ such that

$$\sum_{i=1}^{n_1} \alpha_i(m_i * c_i) + \sum_{i=n_1+1}^{n_1+n_2-1} \alpha_i(c_i * m_i) = 0,$$

then

$$\left(\sum_{i=1}^{n_1} \alpha_i m_i + \sum_{i=n_1+1}^{n_1+n_2-1} \alpha_i c_i\right) * \left(\sum_{i=1}^{n_1} \alpha_i c_i + \sum_{i=n_1+1}^{n_1+n_2-1} \alpha_i m_i\right) = 0 .$$

Therefore,

$$\alpha_1 + \sum_{i=n_1+1}^{n_1+n_2-1} \alpha_i c_i = 0, \qquad \alpha_2 = \cdots \alpha_{n_1} = 0$$

and

$$\sum_{i=1}^{n_1} \alpha_i c_i = 0, \qquad \alpha_{n_1+1} = \cdots = \alpha_{n_1+n_2-1} = 0.$$

Hence also $\alpha_1 = 0$ and these functions are linearly independent.

To see that this collection spans M, let $m \in M$. Then $m = p * q$ where $p \in M_1$ and $q \in M_2$. Letting $c = m(x_1)$ we see that

$$m = (p * c) + (c * q) - (c * c).$$

Hence M has dimension $n_1 + n_2 - 1$.

In general then M has by induction dimension $\sum_{i=1}^{s} n_i - (s - 1)$. It remains to be shown that any nonzero $m \in M$ has no more than $\sum_{i=1}^{s} n_i - s$ sign changes.

Let $m \not\equiv 0$ be in M. Then on any subinterval $[x_{i-1}, x_i]$ for $i = 1, 2,..., s$, either m is identically zero or m has no more than $n_i - 1$ zeros. Certainly m has only finitely many sign changes since otherwise m would have infinitely many sign changes on some subinterval. If we order the points at which m alternates in sign, then to each sign alternation there corresponds at least one distinct zero of m on an interval in which m is not identically zero. There are at most $n_i - 1$ such zeros on any given interval and thus at most $\sum_{i=1}^{s} (n_i - 1)$ on the interval $[a, b]$.

Remarks. (1) It is not known what happens in Theorem 4 if one does not assume each M_i contains the constants.

(2) Another possible converse to Theorem 3, would be the statement that if M is a subspace of $C[a, b]$ and each $M/[x_{i-1}, x_i]$, $i = 1, 2,..., m$ is a Haar set containing 1, then M is a weak Chebyshev set. This converse does not hold as the following example shows. Let $M = \langle 1, m \rangle$ with

$$\begin{aligned} m(x) &= x + 1 & -2 \leqslant x \leqslant -1 \\ &= 0 & -1 \leqslant x \leqslant 1 \\ &= x - 1 & 1 \leqslant x \leqslant 2. \end{aligned}$$

Then M is a Haar set of dimensions 2, 1 and 2, respectively, when restricted to the subintervals $[-2, -1]$, $[-1, 1]$ and $[1, 2]$. But M is not a weak Chebyshev set of dim 2 on $[-2, 2]$. Notice that this M does not arise by "piecing" together Haar sets as in Theorem 4. If the Haar sets in this example were "pieced" together as in Theorem 4, one would obtain a weak Chebyshev set of dimension 3.

Acknowledgment

I thank the referee for many helpful comments about the rewriting of this papər.

References

1. R. C. Jones and L. A. Karlovitz, Equioscillation under nonuniqueness in the approximation of continuous functions, *J. Approximation Theory* **3** (1970), 138–145.
2. S. Karlin and W. J. Studden, "Tchebycheff systems: With Applications in Analysis and Statistics," J. Wiley, New York, 1966.
3. J. R. Rice, Characterization of Chebyshev approximations by splines, *SIAM J. Numer. Anal.* **4** (1967), 557–565.
4. L. Schumaker, Uniform approximation by Tchebycheffian spline functions, Part 1, Fixed knots, Tech. Rept. 768, Mathematics Research Center, Madison, WI, 1967.